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Name.......................................

Reg. No...................................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2021

[November 2020 session for SDE/Private Students]

(CBCSS)

Mathematics

MTH 1C 01—ALGEBRA—I

(2019 Admission onwards)

[Covid instructions are not applicable for PVT/SDE students (November 2020 session)]

Time : Three Hours

Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section / sub part / sub division may be ignored.*
4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions in this part.*
Each question has weightage 1.

1. Do the rotations about one particular point P, together with the identity map, form a subgroup of the group of plane isometries ? Why or why not ?

2. Find the order of (3, 10, 9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

3. Find the order of $5 + \langle 4 \rangle$ in the factor group $\mathbb{Z}_{12} / \langle 4 \rangle$.

4. In the group $G = \mathbb{Z}_{36}$ with $H = \langle 9 \rangle$. List the cosets in G/H, showing the elements in each coset.

5. Show that no group of order 20 is simple.

6. How many different homomorphisms are there of a free group of rank 2 onto $\mathbb{Z}_4$?

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7. Show that $\langle a, b : a^3 = 1, b^2 = 1, ba = a^2 b \rangle$ gives a group of order 6 . Show that it is non

8. Find all zeros of $x^3 + 2x + 2$ in $\mathbb{Z}_7$.

(3 × 1 = 3

Part B

Answer any **six** *questions, choosing* **two** *from each unit.*
Each question has weightage 2.

Unit 1

9. Find all abelian groups, up to isomorphism, of order 360.

10. State and prove the Fundamental Homomorphism Theorem.

11. Let G be a finite group and X a finite G-set. If r is the number of orbits in X under G,

that $r \cdot |G| = \sum_{g \in G} |X_g|$.

Unit 2

12. Let H and K be normal subgroups of a group G with $K \leq H$. Show that $G/H \simeq (G/K)/(H/K)$

13. If G is a finite group and p divides $|G|$, then prove that the number of Sylow p-su

congruent to 1 modulo p and divides $|G|$.

14. If H and K are finite subgroups of a group G, then show that $|HK| = \frac{(|H|)(|K|)}{|H \cap K|}$.

Unit 3

15. Compute the evaluation homomorphism $\phi_4 \left(3x^{106} + 5x^{99} + 2x^{53}\right)$, $F = E = \mathbb{Z}_7$.

16. Show that $f(x) = x + 3x + 2$ viewed in $\mathbb{Z}_5[x]$ is irreducible over $\mathbb{Z}_5$.

17. Let $G = \{e, a, b\}$ be a cyclic group of order 3 with identity element e. Write the

$(2e + 3a + 0b)(4e + 2a + 3b)$ in the group algebra $\mathbb{Z}_5 G$ in the form $re + sa + tb$ for r, s, t

(6 × 2 = 12 we

Part C

Answer any **two** *questions.*
Each question has weightage 5.

(a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1 .

(b) Let H be a subgroup of a group G. Show that left coset multiplication is well defined by the equation $(a\mathrm{H})(b\mathrm{H}) = (ab)\,\mathrm{H}$ if and only if H is a normal subgroup of G.

(a) Let G be the additive group of real numbers. Let the action of $\theta \in \mathrm{G}$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counter clockwise about the origin through θ radians. Let P be a point other than the origin in the plane. Show $\mathbb{R}^2$ is a G-set. Describe geometrically the orbit containing P. Find the group G_p.

(b) Show that the Converse of the Theorem of Lagrange is false.

(a) Prove that every group of prime-power order (that is, every finite p-group) is solvable.

(b) For a prime number p, prove that every group G of order p^2 is abelian.

(a) If F is a field, then prove that every non-constant polynomial $f(x) \in \mathrm{F}[x]$ can be factored in F $[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit (that is, non-zero constant) factors in F.

(b) Show that the polynomial $\Phi_p(x) = \dfrac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + x + 1$ is irreducible over $\mathbb{Q}$ for any prime p.

(2 × 5 = 10 weightage)