C 61262 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MAY 2014

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that the graph of any function $f : \mathbb{R}^n \to$ IR is a level set for some function F $\mathbb{R}$ $\to \mathbb{R}$.

2. Sketch the vector field X (X *(p)* **(P, X** *(p)))* where X *(p)* = *-p.*

3. Sketch the level set f^{-1} (0) and typical values of the vector field ∇f for $p = f^{-1}$ (0) when f $x_2) = x_1^2 -$ $-1.$

4. Let S be an (n – 1) – surface in $\mathbb{R}^n$ given by S = f^{-1} (C) where $f : U \to \mathbb{R}$ (U open in $\mathbb{R}$ is such that V $f(p) \neq 0$ for all p e S). Define the cylinder over S in $\mathbb{R}$ and show that it is an n-surface in $\mathbb{R}$.

5. Describe the spherical image of one sheet of 2-sheeted hyperboloid x? – x2 – $x3$ = **4, x_1 > 0,** oriented by N = $\nabla f / \|\nabla f\|$ where $f(x_1, x_2, x_3) =$ $^2 - x_2^2 - x_3^-$.

6. **Prove** that geodesics have constant speed.

7. Let X and Y be smooth vector fields along the parametrized curve a : I $\to \mathbb{R}$ $^{+1}$. Verify that $[X \cdot Y]' =$ Y + X .

8. Define the Weingarten map $L_p : S_p \to S_p$ (with standard notation).

9. Compute ∇_v X where v E III p, p E R^2 and X is given by X $(x_1, x_2) = (x_1, x_2, x_1, x_2, x_2^2)$, $v = (1,$ **0, 0,1).**

10. Let C be an oriented plane curve and let p E C . Define : a parametrization of a segment of C containing p.

Turn over

11. Find the length of the parametrized curve $\alpha: I \to \mathbb{R}^3$ where I = [-1, 1] and $\alpha(t) = (\cos 3t, \sin 3t, 4t)$.

12. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Define the first and second fundamental forms of S at p.

13. Show that a parametrized 1-surface is simply a regular parametrised curve.

14. Let $\phi: V_1 \to V_2$ and $\psi: V_2 \to \mathbb{R}^k$ be smooth verify the chain rule $d(\psi \circ \phi) = d\psi \circ d\phi$.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Find the integral curve through $p = (a, b)$ of the vector field X given in question 2.

16. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Determine the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $a x_1^2 + 2b x_1 x_2 + c x_2^2 = 1$.

17. Let $S \subset \mathbb{R}^{n+1}$ be an oriented n-surface. Prove that there exists on S exactly two smooth normal vector fields.

18. Choosing an orientation, describe the spherical image of the cylinder $\sum_{i=2}^{n+1} x_i^2 = 1$.

19. Show that a parametrized curve α in the unit sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ is a geodesic iff (if and only if) it is of the form

$$\alpha(t) = (\cos at)\, e_1 + (\sin at)\, e_2$$

for some orthogonal pair of unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$

20. Let S be the n-sphere $\sum_{i=1}^{n+1} x_i^2 = r^2$ oriented by the inward unit normal vector field. Prove that the Weingarten map of S is multiplication by $\frac{1}{r}$ $(r > 0)$

21. "Local parametrization of plane curves are in principle, easy to obtain". Explain the statement and illustrate with an example.

22. Let C be a connected oriented plane curve and let 13: I → C be a unit speed global parametrization of C. Prove that β is either one-to-one or periodic.

23. Find the Gaussian curvature of the Cone

$$x_1^2 + x_2^2 - x_3^2 = 0, \; x_3 > 0 \cdot$$

24. State and prove the inverse function theorem for n-surfaces.

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to R$ be smooth. Let $p \in U$ be a regular point of f and let $C = f(p)$. Then prove the set of all vectors tangent to f^{-1} (C) at p is equal to $[\nabla f(p)]^{\perp}$ (Both set inclusion to be proved).

26. Let S be a compact, connected oriented n-surface in $\mathbb{R}^{n+1}$ Prove that the Gauss map maps S onto the unit sphere S^{n} .

27. Let C be a connected, oriented plane curve. Prove : there exists a global parametrization of C.

28. "Locally n-surfaces and parametrized n-surfaces are the same". State the theorems which lead to the above assertion and outline their proofs.

(2 x 4 = 8 weightage)