C 3545 **(Pages : 4)** **Name..**

Reg. No......................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours **Maximum : 36 Weightage**

Standard notation as in the prescribed text is followed.

Part A

Answer **all** *questions. Each question carries weightage 1.*

1. **Sketch the level sets $f^{-1}(c)$ at the heights indicated**

$$f \quad x_2, x_3) = {}^2 - x_2^2 - x_3^2 ; c = -1, 0.$$

2. **Find and sketch the gradient field of the function** $I(xi, x_2) = \left(x_1^2 - x_2^2\right) / 4$

3. **Show by example that the set of vectors tangent at a point** p **of a level set might be all of** $\mathbb{R}_p^{n+1}$

4. **Show that the set S of all unit vectors at all points of R2 forms a 3-surface in** $\mathbb{R}^4$

5. **Show that if S is a connected n-surface in** $\mathbb{R}^{n+1}$ **and** $g : S \to \mathbb{R}$ **is continuous and takes on on finitely many values, then** g **is constant.**

6. **Describe the spherical image of the paraboloid** $-x_1 + x^2 + x_3^2 = 0$ **(Choose your orientation).**

7. **Show that if** $\alpha : I \to \mathbb{R}^{n+1}$ **is a parametrized curve with constant speed, then** $a(t) \perp a(t)$ **for all** t

8. **Let S be an n-surface in** $\mathbb{R}^{n+1}$**, let** $a : I \to S$ **be a parametrized curve. Let X be a vector tangent to S along a'. Verify that**

$$(f X)' = f X + f X'$$

for all smooth functions f **along a.**

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9. Compute $\nabla_v f$ where $f : \mathbb{R}^2 \to \mathbb{R}$ and $v \in \mathbb{R}^2_p$, $p \in \mathbb{R}$.

$f(x_1, x_2) = x_1^2 - x_2^2$, $v = (1, 1, \cos 0, \sin 0)$.

10. Let C be an oriented plane curve and $p \in C$ with $k(p) \neq 0$. Define the circle of curvature at p.

11. Find the length of the given parametrized curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$, where $\alpha(t) = ($ cos 2t, sin 2t, sin 2t$)$.

12. Let S be an oriented 2-surface in R3 and let $p \in S$. Show that for each v, w $\in S_p$

$L_p(v) \times L_p(w) = k(p)\; v \times w.$

13. Let $\varphi : U_1 \to U_2$ and $\psi : U_2 \to \mathbb{R}$ be smooth. Verify the chain rule $d(\psi \circ \varphi) = d\psi \circ d\varphi$.

14. Show that if $S = f^{-1}(c)$ is an n-surface in $\mathbb{R}^{n+k}$ and $p \in S$, then the tangent space ; to S at p is equal to the kernal of df_p.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions. Each question carries weightage* 2.

Find the integral curve through $p(0, 1)$ of the vector field X on $\mathbb{R}^2$ given by $X(p) = (p, X(p))$ where $X(x_1, x_2) = (-2x_1, x_2)$.

Show that the maximum and minimum values of the function $g(x_1, \ldots, x_{n+1}) = \sum_{j=1}^{n+1} a_{ij} x_i x_j$ on the unit n-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ where (a_{ij}) is a symmetric n x n matrix of real numbers, are the eigenvalues of the matrix (a_{ij}).

17. Let S be an n-surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S and let $p \in S$ Then prove the existence of the maximal integral curve of X through *p*.

18. Show that if the spherical image of a connected n-surface is a single point, then S is contained in an n-plane.

19. For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \to S^2$ be the parametrized curve in the unit sphere S^2 from the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$, defined by $\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t)$. Let $v = (p, 1, 0, 0) \in S_p$. Then compute $P_{\alpha_\theta}(v)$.

20. Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field N. Let $p \in S$ and $v \in S_p$. Let $\alpha : I \to S$ be a parametrized curve with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$. Then prove that $\ddot{\alpha}(t_0) \cdot N(p) = L_p(v) \cdot v$.

21. Let $g : I \to \mathbb{R}$ be a smooth function and let C denote the graph of g. Show that the curvature of C at the point $(t, g(t))$ is $g''(t) / (1 + g'(t)^2)^{3/2}$ for an appropriate choice of orientation.

22. Find the Gaussian curvature of the ellipsoid $\frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{x_3^2}{9} = 1$

23. Show that the Weingarten map at each point of a parametrized n-surface in $\mathbb{R}^{n+1}$ is self-adjoint.

24. State and prove inverse function theorem for n-surfaces.

(7 x 2 = 14 weightage)

Part C

Answer any two questions.
Each question carries weightage 4.

25. Let S be a compact, connected oriented n-surface in $\mathbb{R}^{n+1}$. Prove that the Gauss map maps S onto the unit n-sphere S^n.

Turn over

26. Let C be a connected, oriented plane curve and let 0 : I C be a unit speed global parametrization C. Then prove that β is either one-to-one or periodic. Further show that 0 is periodic iff C is compact.

27. Let S be a compact oriented n-surface in $\mathbb{R}^{n+1}$. Prove : There exists a point p E S such that the second fundamental form at p is definite.

28. Let S be an n-surface in $\mathbb{R}^{n+1}$ and let f $S \to \mathbb{R}$. Suppose that f o g is smooth for each level parametrization, $\varphi : U \to S$. Then prove that f is smooth.

(2 x 4 = 8 weightage)

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