C 82512 (Pages : 3) Name

Reg. No..................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that the graph of any function f $\longrightarrow \mathbb{R}$ is a level set for some function F $: \mathbb{R}^{+1} \to \mathbb{R}$.

2. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 - x_2$.

3. Let $f : U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f. Show that :

   $\left(\frac{d}{dt}\right)(f \quad d)(t) = \| \nabla f(\alpha(t)) \|^2$ for all $t \in I$.

4. Sketch the cylinder over the graph of $f(x) = \cos x$

5. Show that the two orientations on the unit n-sphere $x_1^2 + \ldots + 4_{+1} = 1$ are given by :

   $N_1(p) = (p, p)$ and $N2(p) = (-p, p)$.

6. Prove that geodesics have constant speed.

7. Let S be an n-surface in $\mathbb{R}^{n+1}$, let a $: I \to S$ be a parametrized curve and let X and Y be vector fields tangent to S along a. Verify that (X + Y) = X +

8. Compute V (1) where $f : \mathbb{R}^{+1} \quad \mathbb{R}, p \in {}^{n+1}, v \in \mathbb{R}_p$ where $f(x_1, x_2, x_3) = x_1, x_2, x_3^2$ and $v = (1,1,1,a,b,c)$ $(n = 2)$.

9. Find a global parametrization of the plane curve $x_1^2 + \frac{x_2}{4} = 1$. (You may choose the orientation).

Turn over

10. Find the length of the parametrized curve a : [0,2m] $\rightarrow R^4$ given by $\alpha(t) = (\cos t, \sin t, \cos t, \sin t)$.

11. Let S $\mathbb{R}^n$ be an oriented n-surface, let p **E S**. Define the second fundamental form of S at p.

12. Let U be an open set in $\mathbb{R}$, let $\varphi : U \rightarrow \mathbb{R}^m$ be a smooth map, let $d\varphi$ be the differential of 9.

Prove that the restriction $d\varphi_p$ of $d\varphi$ to $\mathbb{R}^n_p$ is a linear map $d\varphi_p : \mathbb{R}^n_p \quad \mathbb{R}^m_{\varphi(p)}$

13. Let Q : U_1 **U**$_2$ and $\psi : U_2$ -3 I be smooth where U_1 c $\mathbb{R}'$ and $U_2 \subset \mathbb{R}^m$. Verify the chain rule $d(\psi \circ \varphi) = d\psi \circ d\varphi$.

14. Let S be an n-surface in $\mathbb{R}^n$ ' $(k . 1)$. Let p E S. Define the tangent space S_p at p.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Find the integral curve through $p = (a, b)$ of the vector field X on $\mathbb{R}$ ^ given by $X(p) = (p, X(p))$, where X $(x_1, x_2) = (x_2, x_1)$.

16. Sketch the tangent space at a typical point of the level set f ˉ (1) where $f(x_1, x_2, x_3) = x_1^2 + x2 + x_3^2$.

17. Show that the set S of all unit vectors at all points of 1R^2 forms a 3-surface in R^4.

18. Show that the spherical image of an n-surface with orientation N is the reflection through the origin of the spherical image of the same n-surface with orientation – N.

19. Prove that, in an n-phase, parallel transport is path independent.

20. Let S be the unit n-sphere $\sum_{i=1}^{n+1} x2 = 1$ oriented by the outward unit normal vector field. Prove that the Weingarten map of S is multiplication by —1.

21. Let $\alpha(t) = (x(t), y(t))$ (t E **I)** be a local parametrization of the plane curve C. Show that :

$$r \circ \alpha = (x' y'' — y' x)/(x'^2 / y'^2)^{3/2}.$$

22. Let S be the ellipsoid $\left(x_1^2 / a^2 + \quad / b^2\right) + \quad c2 = 1$. Find the Gaussian curvature of S. $(abc \neq 0)$.

23. Show that the Weingarten map at each point of a parametrized n-surface is self-adjoint.

24. State and prove the Inverse Function Theorem for n-surfaces.

(7 x 2 = 14 weightage)

Part C

Answer any two questions.
Each question carries 4 *weightage.*

25. Let U be an open set in $\mathbb{R}^{\ +1}$ and let $f: U \to$ IR be smooth. Let p E U be a regular point of f and let $c = f(p)$. Then prove : the set of all vectors tangent to $f(c)$ at p is equal to [v $f(p)$] .

26. Let S be an n-surface in $\mathbb{R}^{\ +1}$, let p E S and let v E S_p. Then prove there exists an open interval I containing 0 and a geodesic a : I S such that :

(i) a (0) = p and $\dot{\alpha}(0) =$

(ii) If [3: I -p S is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) =$ v, then I c I and $\beta(t) =$ a(t) for all t E .

27. Let ii be the 1-form on $R^2 - \{0\}$ defined by :

$$11 = \frac{x_2}{x_1^2 + X2^2} d_1 \quad \frac{x}{x \;\; + X2^2} dx_2$$

Then prove that for a : [a, $b] \to R^2 - \{0\}$, any piecewise smooth closed parametrized curve in $R^2 - \{0\}$ J hl = 2 for some integer K.

28. Let $\quad \to \mathbb{R}^{\ +1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let p E U . Then show that there exists an open set U_1 c U about p such that φ (U1) is an n-surface in $\mathbb{R}^{n+1}$.

(2 x 4 = 8 weightage)