C 42502 (Pages : 3) Name...

Reg. No.................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2013

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum 36 Weightage

Part A

Answer all *questions.*
Each question carries 1 weightage.

1. For the given function f, sketch level sets $f^{-1}(c)$ at the heights indicated.

   f $x_2) =$ $-$ $c = -1, 0, 1.$

2. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1^2 +$

3. Show by example that the set of vectors tangent at a point p of a level set might be all of $\mathbb{R}_p^{+}$.

4. Sketch the surface of revolution obtained by rotating c about the x_1 axis where c is the curve $-\ ^2 + ^2 = 1, x_2 > 0.$

5. Show that the plane R^2 is connected.

6. Describe the spherical image of the cone.

   $-\ ^2 + x_2^2 \quad x_3^2 = 0, x_1 > 0.$

   (the surface is oriented by where f is the function $- x_1^2 + x_2 + x_3$).

7. Define a geodesic on an n-surface $S \subset \mathbb{R}^{n+1}$. If S contains a straight line segment, prove that segment is a geodesic.

8. Let S be an n-surface in $\mathbb{R}^{n+1}$, let a : I → S be a parametrised curve is S ; X, Y smooth vector fields tangent to S along a. Then prove that (X . $= X' \cdot Y + X \cdot Y'$

Turn over

9. Compute the derivative ∇_v (X) where v E $\mathbb{R}^n_p$, p E R^2 given by X (x1, x2 Xi, X2 – X2, Xi), v = (cos 0, sin 0 – sin 0, cos 0).

10. Find global parametrisation of x_2 – = 0. (You may choose the orientation).

11. Find the length of the parametrised curve $a : I \to R^4$ where $a(t)$ = (cos t, sin t, cos t, sin t), $I = [0, 2\pi]$.

12. Let S be an oriented n - surface in $\mathbb{R}^{n+1}$ Define the first and second fundamental forms of S at p E S.

13. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let p E S. Give a formula for computing K(p), the Gauss-Kronecker curvature.

14. Let U be an open set in $\mathbb{R}^n$ and let $Q : U \to \mathbb{R}$ be a smooth map. Define : dQ, the differential of Q.

(14 x 1 = 14 weightage)

Part B

Answer any seven *questions.*
Each question carries 2 *weightage.*

15. Let X be the vector field on R^2, X $(p) = (p, X(p))$ where X (x_1, x_2) = $\left(-2x_2, \frac{\cdot}{2}\right)$. Find the integral curve of X through $p = (1,1)$.

16. Show that the maximum and minimum values of the function $g(x_1, \ldots) = \sum_{i=1}^{n+1} a_{ij} x_i x_j$ on the unit n-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$, where (ad) is a symmetric n real matrix, are eigen values of the matrix (a_y).

17. Show that the two orientations on the n-sphere $\sum_{i=1}^{n+1} x_i^2 = r^2$ of radius $r > 0$ are given by $N_1(p) = (p, p\ 1\ r)$ and **N2** $(p) = (p, -p\ I\ r)$.

18. Describe the spherical image of the parabola $-x_1 + x2 = 0$ (orientation left to your choice).

• 19. Let S denote the cylinder x_1 + x2 =1 in $\mathbb{R}$ Show that a is a geodesic of S iff a is of the form a *(t)* = (cos *(at + b),* sin *(at + b),* [Ct + Cd)), for some a, *b, c, d* E R.

20. Let a : [0, π] s^2 be the half great circle in s^2 running from *p* = (0, 0, **1) to** *q* = (0, 0, – **1)** defined by a *(t)* = (sin *t,* 0, cos *t).* Let v = *(p,* 1, 0, 0) E **S** *p*, show that P_a (v) = *(q,*— 0, 0).

21. Choosing your own orientation, compute the Weingarten map for the circular cylinder $x_2^2 + x_3^2 = 1$ in R^3

22. Let C be a connected oriented plane curve and let 13 : I → C be a unit speed global parametrization of C. Then prove that 13 is periodic if and only if C is compact.

23. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+}$. Then, prove that the Gauss - Kronecker curvature of S at *p* is non - zero for all *p* E S if and only if second fundamental form s_p of S at *p* is definite for all *p* E S.

24. Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let

$\pi(S) = \{v \in \mathbb{R}_p^{n+1} \subseteq \mathbb{R}^{2n+2}$: *p* E S and v • N (p) Prove that T (S) is a 2n— surface in $\mathbb{R}^{n+2}$.

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Let U be an open set in $\mathbb{R}^n$ +1 and let *f* = U -3111 be smooth. Let *p* E U be a regular point off and let *c* = *f (p).* Then prove that the set of all vectors tangent to f^{-1} *(c)* at *p* is equal to [V *f* (*p*)] (Both set inclusion to be proved)

26. Let S be an n-surface in $\mathbb{R}$ +1, let *p* E S and let v E S. Then prove the existence and 'uniqueness' of the maximal geodesic in S passing through *p* with initial velocity v.

27. Prove that the Weingarten map Lp is self-adjoint.

28. Let S be a compact oriented n-surface in $\mathbb{R}^n$ +1 Prove there exists a point *p* on S such that the second fundamental form at *p* is definite.

(2 x 4 = 8 weightage)