C 61261　　　　**(Pages : 3)**　　　　**Name.............**

Reg. No.................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MAY 2014

(CUCSS)

Mathematics

MT 4C 15—FUNCTIONAL ANALYSIS—II

Time : Three Hours　　　　Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Each question carries a weightage of 1.

1. Show that a continuous map on a metric space is a closed map. Is the converse true ? Justify your answer.
2. Define eigen value, and approximate eigen value of a bounded operator A on a Banach space X. Show that every eigen value of A is an approximate eigen value of A.
3. Define Fredholm integral operator on X = C ([0, 1]).
4. Show that if A is an invertible bounded linear operator on a Banach space X .over K then $\sigma(A^{-1}) = \{K^{-1} \in K : K \in \sigma(A)\}$
5. Let X and Y be normed spaces and F $\in$ **BL** (X, Y). Show that if y' $\in$ Y', then $\|F'(y')\| \leq \|F\| \cdot \|y'\|$.
6. Show that every reflexive normed space is a Banach space.
7. Give an example of a continuous map on a normed space which is not compact.
8. Let H be a Hilbert space over K and y $\in$ H be fined. Show that $f : H \to K$ defined by $f(x) = < x, y >$ for x $\in$ H is a continuous linear functional and that $\|f\| = \|y\|$
9. Define weak convergence of a sequence (x_n) in a Hilbert space H. Is it true that if a sequence (x_n) in H is convergent in H then (x_n) is weak convergent in H ? Justify your answer.
10. Show that if A is a linear operator on a Hilbert space H which is continuous at 0, then A is bounded.
11. Show the {$u_1, u_2, \ldots$} is an orthonormal basis for a Hilbert space H, then each A $\in$ BL (H) is defined by the matrix $(< A(u_j), u_i >)$ with respect to this basis.

Turn over

12. Let H be a Hilbert space and A s BL (H). Determine the relation between A* and A', where A* and A' are respectively the adjoint and transpose of A.

13. Give an example of a normal operator which is neither unitary nor self-adjoint.

14. Let H be a Hilbert space and A, B c BL (H) with A self-adjoint. Show that AB = 0 iff R (A) 1 R (B).

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries a weightage of 2.

15. Let $\| \|'$ be a complete norm on C b]) such that if $\|x_n - x\|' \to 0$, then $x_n(t) \to x(t)$ for every $t \in [a, b]$. Show that $\| \|'$ is equivalent to, the sup norm on C .

16. Let X be a non-zero Banach space over C and A c BL (X). Show that a (A) is non-empty.

17. Let X be a normed space. Show that if X' is separable then X is separable.

18. Show that every closed subspace of a reflexive normed space is reflexive.

19. Let X be a Banach space and P c BL (X) be a projection. Show that P is compact iff P is of finite rank.

20. Let H be a Hilbert space. Show that there is an inner product $< , >'$ on H' such that $< f, f > = \|f\|^2$ for every $f \in H'$.

21. Let H be a Hilbert space and A ε BL (H). Show that there is a unique B c BL (H) such that for all x, y H,

$$< A(x), y > = < x, B(y) >.$$

22. Let H be a Hilbert space and A ε BL (H). Show that 4 is unitary iff $\|A(x)\|$ =114 for all x c H and A is surjective.

23. Let A c BL (H) and for n = 1, , let A_n be a compact operator on H. Show that if. $\|A_n - A\| \to 0$, then A is compact.

24. Let A be a compact self-adjoint operator on a Hilbert space H. Show that A is a positive operator iff every eigenvalue of A is non-negative.

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 4.

25. State and prove closed graph theorem.

26. Let $1 \leq p, q \leq \infty$ and $\frac{1}{p}$ Show that the dual of K^n with the norm is linearly isometric to K^n with the norm $\| \|_q$.

27. Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Show that there is a unique continuous linear functional f on H such that $f/_G = g$ and lin $= \|g\|$.

28. Let A be a non-zero compact self-adjoint operator on a Hilbert space H over K. Show that there exists a finite or infinite sequence (s_n) of non-zero real numbers with and an orthonormal set $u_2, \ldots\}$ in H such that

A (x) = $s_n < x, u_n > u_n, x \in H$. Further show that if the set $u_2, \ldots\}$ is infinite, then

s_n 0 as n $\rightarrow$

(2 x 4 = 8 weightage)