Name.................................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathemathics

MT 4C 15—FUNCTIONAL ANALYSIS—II

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. **Let P be a bounded linear map from a normed space X into itself. If P is a projection, then prove that the range space $\mathcal{R}(P)$ and the null space *Z (2)* are closed in X.**
2. **Give an example of a closed linear map between Banach spaces which is not open.**
3. **Let X and Y be Banach spaces and F be a bounded linear, bijective map from X to Y. Using closed graph theorem, prove that F^{-1} is a bounded linear map from Y to X.**
4. **Let X be Banach space. Prove that the set of all bounded invertible operators on X is an open subset of the set of all bounded operators on X.**
5. **If A is an invertible bounded operator on a normed space X, then prove that a $(A^{-1}) = \; : k \in \sigma(A)\}$ where $\sigma(A)$ is the spectrum of A.**
6. **If X is a finite dimensional normed space, then prove that its eigen spectrum, approximate eigen spectrum and the spectrum are the same.**
7. **Define reflexive normed spaces and give an example of it.**
8. **Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. Find $E^{\perp}$ and prove that it is a closed subspace of the Hilbert space R^2.**
9. **Does projection theorem hold for incomplete inner product spaces ? Justify your answer.**
10. **Prove that in a finite dimensional Hilbert space weak convergent sequences are convergent.**
11. **Let H be a Hilbert space and let A be a bounded operator on H. Prove that $\|A\| = \|A\|$.**
12. **Let H be a Hilbert space and let A be a bounded operator on H. Prove that the closure of $\mathcal{R}(A)$ equals $\mathcal{Z}(A)^{\perp}$, where $\mathcal{R}(A)$ is the range space of A^* and $\mathcal{Z}(A)$ is the null space A.**

13. Let A be a normal operator on the Hilbert space H. If k is an eigen value of A, then prove that $\bar{k}$ is an eigven value of A^* and the eigen vector of A corresponding to k is an eigen vector of A^* corresponding to $\bar{k}$.

14. Give an example of a Hilbert-Schmidt operator on the Hilbert space 1^2.

(14 x 1 = **14** weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Let X be a normed space over C and let $f: X \to C$ be a linear map. Prove that f is closed if and only if it is continuous.

16. Let X be a Banach space and let A be a bounded linear map on X. Prove that A is invertible if and only if A is bounded below and the range of A is dense in X.

17. Let X be a Banach space over C and let A be a bounded operator on X. Prove that $\sigma(A)$ is a compact subet of C.

18. Let X and Y be normed spaces and let F be a bounded linear map from X to Y. Prove that the transpose F' of F is a bounded linear map from Y to X and $\|F'\| =$

19. Prove that the dual X' of a reflexive normed space X is reflexive.

20. Let H be a Hilbert space and let F be a non-empty closed subspace of H. Prove that H = F

21. Let H be a Hilbert space. Prove that the set of all normal operators on H is a closed subet of the set **of all bounded operators on H.**

22. Let H be a Hilbert space and A be an operate on. H. Prove that the spectrum of A is contained in the closure of the numerical range $\omega(A)$.

23. Let A be a self adjoint operator on a finite dimensional Hilbert space H. Prove that every root of the characteristic polynomial of A is real.

24. Let A be a compact operator on a non-zero Hilbert space H. Prove that every non-zero approximate eigenvalue of A is an eigenvalue of A.

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries **4** *weightage.*

25. Let X be **a non-zero Banach space over C and A be a bounded linear map on X. Prove that the spectrum of A is non-empty and its spectral radius r_σ is :**

$$r_\sigma = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

26. State and prove Riesz representation theorem for continuous linear functionals on a Hilbert space.

27. Let H be a Hilbert space over C and let A be a bounded linear map on H. Prove that :

    (1) ***k*** is a special value of A if and only if $\bar{k}$ is a spectral value of A*.

    (ii) $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A)$ … $: k \in \sigma_e(A^*)\}$, where $\sigma(A)$, $\sigma_e(A)$ and $\sigma_a(A)$ are the spectrum, eigen spectrum and approximate eigen spectrum of A.

28. Let A be a non-zero compact self adjoint operator on a Hilbert space H over C. Prove that there exist an infinite sequence $\{s_n\}$ of non-zero real numbers with $|s_1| \geq |s_2|$ … and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that

$$A(x) \sum s_n \langle x, u_n \rangle u_n.$$

(2 x 4 = 8 weightage)