3547 (Pages : 3) Name..

Reg. No......................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CUCSS)

Mathematics

MT 4E 02—ALGEBRAIC NUMBER THEORY

Time : Three Hours **Maximum : 36 Weightage**

Standard notation as in prescribed text is followed.

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. **Express $t_1^4 + 4$ in terms of elementary symmetric polynomials (n = 2).**

2. **Find the order of the group G/H where G is a free abelian group with basis x, y, z and H is generated by – 2x, x + y, y + z.**

3. **Find 0 such that Q(0) = $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$.**

4. **Show that an algebraic number is an algebraic integer if and 'only if (iff) its minimal polynomial over Q has coefficients in Z .**

5. **Let K = $\mathbb{Q}(\zeta)$ where = e2 n ' /5 . Calculate N (a) and T_K (a) for a =**

6. **Let x and y be non-zero elements of a domain D. Prove that $x \mid y$ if $(x) \supset (y)$.**

7. **Find a ring which is not noetherian.**

8. **Is 10 = (3 + i) x (3 – i) = 2 x 5 an example of non-unique factorization in $\mathbb{Z}[i]$? Give reasons for your answer.**

9. **True *or* False ?**

   A fractional ideal of is a finitely generated $\mathfrak{D}$-submodule of K.

10. **Prove : If σ is a proper ideal of the ring of integers of the number field K_1 then G $^{-}$ properly contains D .**

11. **State Minkowskis theorem.**

Turn over

12. Show that the quotient group is $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group *S*.

13. Sketch the lattice R2 generated by $(-1, 2)$ and $(2, 2)$ and a fundamental domain for the lattice.

14. Let d be a squarefree positive integer and let $K = \mathbb{Q}(\sqrt{d})$. Calculate a K

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries weightage 2.

15. Let G be a finitely generated abelian group with no non-zero elements of finite order. Prove that G must be a free group.

16. Prove that the set of algebraic numbers is a subfield of the complex field C.

17. Let K = Q(0) be a number field. Prove : If all k-conjugates of 0 are real, then the discriminant of any basis is positive.

18. Let K be a number field of degree n-prove that the 2, the ring of integers of K, is a free abelian group of rank n.

19. Let d be a squarefree rational integer with $d \equiv 1 \pmod 4$. Then prove that $\mathbb{Z}[\sqrt{d}]$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$.

20. Prove that the group of units of $\mathbb{Q}(\sqrt{-3})$ is the group $\{\pm 1, \pm w, \pm w^2\}$ where $w = e$ [illegible]

21. Prove that an integral domain 2 is noetherian if $\mathfrak{D}$ satisfies the maximal condition.

22. Prove that an ring of integers of $\mathbb{Q}(\sqrt{-5})$ is not a unique factorization domain.

23. If x, y, z are integers such that $x^2 + y^2 = z^2$, prove that atleast one of *x, y, z* is a multiple of 3.

24. Prove : If $a_1, \ldots, a_n$ is a basis of the number field K over Q, then $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$ are linearly independent over $\mathbb{R}$.

7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage **4.**

. Let K be a number field. Then prove that there is an algebraic integer $\theta \in k$ such that $k = \mathbb{Q}(\theta)$.

26. Let $\zeta = e^{2\pi i/p}$ where p is an odd prime. Prove that $\mathbb{Z}[\zeta]$ is the ring of integers of Q {] .

27. Let O be a domain in which factorization into irreducibles is possible. Prove that factorization into irreducibles is unique iff every irreducible is prime.

28. Prove that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

(2 x 4 = 8 weightage)