

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS—I

Time : Three Hours

Maximum : 36 Weightage

Part A

*Answer all questions.**Each question carries 1 weightage.*

1. Prove or disprove : A sequence in a metric space is bounded in X iff it is Cauchy.
2. Show that the metric space \mathcal{R} is complete.
3. Define n^{th} Dirichlet Kernel D_n and evaluate $\int D_n(t) dt$.
4. State Riesz's lemma.
5. Let Y be a subspace of a normed space X . Show that $Y^0 = \emptyset$ iff $Y = X$.
6. Illustrate with an example that a linear map on a linear space X may be continuous with respect to some norm on X , but discontinuous with respect to another norm on X .
7. Let X be an inner product space and $x \in X$. Show that $(x, y) = 0$ for all $y \in X$ iff $x = 0$.
8. Show that if E and F are closed subsets of a Hilbert space H and $E \perp F$, then $E + F$ is closed in H .
9. Let X be an inner product space. Show that if $E \subset X$ is convex, then there exists at most one approximation from E to any $x \in X$.
10. Let X be a normed space over K . Let $\{a_1, a_2, \dots, a_n\}$ be a linearly independent set in X . Show that there are f_1, f_2, \dots, f_n in X' such that $f_j(a_i) = \delta_{ij}, 1 \leq j \leq n$.

11. With usual notations, show that $C_r(T)$ is not closed in $C_o(T)$.
12. Define **Schauder** basis for a **normed** space X and show that if there is a **Schauder** basis for a **normed** space X , then X must be separable.
13. Show that the linear space C_{oo} cannot be a **Banach** space in any norm.
14. State Uniform **boundedness** principle and interpret it geometrically.

(14 x 1 = 14 weightage)

Part B

Answer **any seven** questions.
Each question carries 2 *weightage*.

15. Show that the set of all polynomials in one variable is dense in $C([a, b])$ with the sup metric.
16. Show that the metric space $L^p([a, b])$ is separable for $1 < p < \infty$, but the metric space $L^1([a, b])$ is not separable.
17. Show that for all $x \in \mathbb{K}^n$;

and

$$\|x\|_1 \leq \sqrt{n} \|x\|_2 \leq n \|x\|_\infty.$$

18. Let $X = \mathbb{K}^3$ for $x = (x(1), x(2), x(3)) \in X$, let $\|x\| = (|x(1)|^3 + |x(2)|^3 + |x(3)|^3)^{1/3}$. Show that $\|\cdot\|$ is a norm on \mathbb{K}^3 .
19. Let X and Y be **normed** spaces and Z be a closed subspace of X . Show that if $F \in BL(X/Z, Y)$ and we let $F(x) = F(x + z)$ for $x \in X$, then $F \in BL(X, Y)$ and $\|F\| = \|F\|$.

1. Show that if a non-zero Hilbert space H over \mathbb{K} has a countable **orthonormal** basis then H is **linearly isometric to \mathbb{K}^n** for some n , or to l^2 .

Let E be a non-empty closed convex sub-set of a Hilbert space H . Show for each $x \in H$, there exists a unique best approximation from E to x .

22. Let $X = K^2$ with the norm $\| \cdot \|$. Consider $Y = \{(x(1), x(2)) \in X : x(1) = x(2)\}$, and define $g \in Y$ by $g(x(1), x(2)) = x(1)$.
- Show that the **Hahn-Banach** extensions of g to X are given by :
- $$f(x(1), x(2)) = tx(1) + (1-t)x(2), \text{ where } t \in [0,1] \text{ is fixed.}$$
23. Show that a **normed** space X is a **Banach** space iff every absolutely **summable** series of elements in X is **summable** in X .
24. Show that a subset E of a **normed** space X is bounded in X iff $f(E)$ is bounded in K for every $f \in X'$
- (7 x 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries 4 weightage.*

25. Show that every finite dimensional subspace of a **normed** space X is closed in X .
26. Let $\{u_n\}$ be an **orthonormal** set in a Hilbert space H . Show that $\{u_n\}$ is an **orthonormal** basis for H iff space $\{u_n\}$ is dense in H .
27. State and prove **Hahn-Banach** separation theorem.
28. Let X be a **normed** space and Y be a closed subspace of X . Show that X is a **Banach space** iff Y and X/Y are Banach spaces in the induced norm and the quotient norm, respectively.
- (2 x 4 = 8 weightage)