**D 31617** (Pages : 2) Name

Reg. No..................

**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2012**

(Non-CUCSS)

Mathematics

Paper XII—FUNCTIONAL ANALYSIS—I

(2002 Admissions)

Time : Three Hours Maximum : 80 Marks

**Part A**

*Answer* **all** *questions.*
*Each question carries* 4 *marks.*

I. (a) Define Cauchy sequence and show that every Cauchy sequence is bounded. Is the converse true ? Justify your answer.

(b) Prove that among all the normed spaces $L^p([0,1]); 1 \le p \le \infty$, only the space $L^2([0,1])$ is an inner product space.

(c) Show that the linear space $C_{00}$ cannot be a Banach space in any norm.

(d) Let $\| \ \|'$ be a complete norm on $C([a, b])$ such that if $\|x_n - x\|' \to 0$, then $x_n(t) \to x(t)$ for every $t \in [a, b]$. Show that $\| \ \|'$ is equivalent to the sup norm on $C([a, b])$.

(4 x 4 = 16 marks)

**Part B**

*Answer any* **four** *questions without omitting any unit.*
*Each question carries* 16 *marks.*

UNIT I

II. (a) Show that the intersection of a finite number of dense open subsets of a metric space X is dense in X.

(b) Let T be a compact metric space and $E \subset C(T)$. Suppose that E is bounded and equicontinuous at each $t \in T$. Show that E is totally bounded in the sup metric on C (T).

III. (a) Show that for $1 \le p < \infty$, the metric space $L^p(E)$ is complete for any measurable subset E of R.

(b) Let $x$ be a continuous k-valued function on $[-\pi, \pi]$ such that $x(\pi) = x(-\pi)$. Show that the sequence of arithmetic means of the partial sums of the Fourier series of x converges to x uniformly on $[-\pi, \pi]$.

**Turn over**

IV. (a) State and prove Riesz Lemma.

(b) Let X be a normed space. Show that X is finite dimensional if every closed and hour
subset of X is compact.

UNIT II

V. (a) Show that every linear map on a finite dimensional normed space is continuous.

(b) Show that a linear functional *f* in a normed space X is continuous iff the zero space $Z(f)$ is closed in X.

(c) Let X be a normed space and $P \in BL(X)$ satisfy $P^2 = P$. Show that $\|P\| = 0$ or PI( 1.

VI. (a) Let X be a normed space over *k*, Y be a subspace of X and $g \in Y'$ Show that there is some $f \in X'$ such that $f/_{Y} = g$ and $\|f\| = \|g\|$.

(b) Show that there exists a linear functional *f* on *r* such that $\|f\|$ ' = *f (a)* and *f (x))* = *f* (x) for all x E r where *a* = ...) and $T(x)(j) = x(j+1)$ for *j* =1, 2,...

VII. (a) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff $x \in H$ and $(x, u_\alpha) = 0$ for all a, then $x = 0$.

(b) Let $H = L^2([0,1])$. Show that $\{1, \sqrt{2}\cos \pi t, \sqrt{2}\cos 2\pi t, \ldots\}$ is an orthonormal basis for H.

UNIT III

VIII. (a) Show that a normed space Xis a Banach space iff every absolutely summable series of elements in X is summable in X.

(b) Show that every normed can be embedded as a dense subspace of a Banach space.

IX. (a) Let X be a normed space and E be a subset of X. Show that E is bounded in X if *f* (E) is bounded in *h* for every *f* E X'.

(b) Let 1 5_ *p S cc* and $X = C_{00}$ with the norm 11 $\|_p$. For $n = 1, 2, \ldots$, let $f_n(x) = nx(n)$; $x \in X$.

Show that $f_n(x) \to 0$ for every x E X, but $\|f_n\| \to \infty$.

X. (a) State and prove closed graph theorem.

(b) Show that the closed graph theorem may not hold for normed spaces.

(4 x 16 = 64 marks)