D 31619 (Pages : 3) Name

Reg. No........ ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2012

(Non-CUCSS)

Mathematics

Paper XIV—DIFFERENTIAL GEOMETRY

(2002 admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1 (a) Show that the graph of any function f → R is a level set for some function $f : \mathbb{R}^{n+1}$

(b) Find the integral curve through the point (1, 1) of the vector field X on $\mathbb{R}^2$ with associated function X $\mathbb{R}$ $\mathbb{R}$ given by $X(x_1, x_2) = (x_2, x_1)$.

(c) Show that covariant differentiation of vector fields has the following property

(X • Y) = (DA) + X(p) · S ,Y fields on

(d) Let S be on oriented 2-surface $\mathbb{R}^3$ and let E S Show that fur each

v, *to* E S, $L_p(v) \times L_p(w) = K(p)$ v x w

(4 x 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

2. (a) Let X he a smooth vector field on an open set U $\mathbb{R}^{n+1}$ and let p E U. Prove the existence and uniqueness of the maximal integral curve of X through *p*.

(b) A smooth vector field X on an open set U of $\mathbb{R}^{n+1}$ is said to be complete if for each *p* E U, the maximal integral curve of X through *p* has domain equal to $\mathbb{R}$. Determine which of the following vector fields are complete :

(i) X $(x_1, x_2) = (x_1, x_2, 1, 0)$, U = R^2.

(ii) X $(x_1, x_2) = (x_1, x_2, 1 + $ ‹ 0), = $\mathbb{R}^2$.

Turn over

3. (a) Let p E U be an open set in $\mathbb{R}^{n+1}$ and $f : U \to R$ be smooth. Let p E U be a regular point of f with $f(n) = c$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]$

(b) Show that the set S of all unit vectors at all points of R^2 forms a 3-surface in R^4.

4. Let S be a compact oriented n-surface in . Prove that the Gauss map, maps S onto the unit sphere S^n.

Unit II

5. (a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let a S be a parametrized curve in 5, let to E 1, and let V E $S_{a(t_0)}$ Then prove that there exists a unique vector field V, tangent S along a, which is parallel and has V $(t_o) = v$.

(b) Show that if a S is a geodesic in an n-surface and *if* $f3 = \alpha o h$ is a reparametrization of a (with h → I), then β is a geodesic in S if and only if there exist $a, b \in$ R such that $h(t) = at + b$ for all t

6. (a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and p E S, v E S_p. Define the Weingarten map L_p of S at p. Choosing your own orientation, compute the Weingarten map for the circular cylinder x22 +x3 =1.

(b) Let $a(+) = (x(+), y(+))$ $(t$ E I) be a local parametrization of the oriented plane curve Show that $K \circ \alpha = (x'y'' - y'x'')/(x'^2 + y'^2)^{3/2}$.

7. Let C be a oriented plane curve: Prove that C has a global parametrization.

Unit III

8. (a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let V be a unit vector in S_p p E . Then prove : there exists an open set V c $\mathbb{R}^{n+1}$ containing p such that S n $N(v)$ n V is a plane curve. Further, the curvature at p of this curve (suitably oriented) equals the normal curvature $k(v)$•

(b) Find he Gaussian curvature K S → R where S is the cone xi + x22 $- x_3^2$ O,; > 0.

9. (a) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let p E S. Then prove that there exists an open set V about p in $\mathbb{R}^{n+1}$ and a parametrized n-surface $\varphi : U \to \mathbb{R}^{n+1}$ such that φ is a one-to-one map from U onto V ∩ S

(b) Show that the Weingarten map at each point of a parametrized n-surface is self-adjoint.

10. (a) Let S be an n-surface in $\mathbb{R}^{n+}$ and let $f : S \to R^R$ be such that $f \circ \varphi$ is smooth for each local parametrization $\varphi : \quad \to S$ Prove that f is smooth.

(b) Let S be a compact, connected oriented n-surface in $\mathbb{R}^{n+}$ whose Gauss-Kronecker curvature is nowhere zero. Then prove that the Gauss map N $\quad S^n$ is a diffeomorphism.

(4 x 16 = 64 marks)