

SECOND SEMESTER M. S c. DEGREE EXAMINATION, JUNE 2019

(CUCSS)

Mathematics

MT 2C 07—ALGEBRA—II

(2016 Admissions)

Time : Three Hours

Maximum : 36 Weightage

Part A*Answer all questions.**Each question carries 1 weightage.*

1. Verify whether $\mathbb{Z}/4\mathbb{Z}$ is an integral domain.
2. Find $\text{irr}(x^2 + 1; \mathbb{Q})$.
3. Show that $\mathbb{Q}(\sqrt{-4}, \sqrt{-1})$ is a simple extension.
4. Find the degree $[\mathbb{Q}(\omega, \omega^2) : \mathbb{Q}]$ where ω is a non-real cube root of unity.
5. Let $a = 1 + 2i$ be a complex number. Show that $\mathbb{R}(a) = \mathbb{C}$.
6. Find all automorphisms of $\mathbb{Q}(\sqrt{r})$.
7. Is f constructible. Justify your answer.
8. Verify whether $F = \{0, 1, a, 1 + a\}$ is a subfield of \mathbb{Z}_2 where \mathbb{Z}_2 is an algebraic closure of \mathbb{Z}_2 containing a zero a of $x^2 + x + 1$.
9. Let α be an automorphism of $\mathbb{Q}(\sqrt[3]{1}, \sqrt[3]{\beta})$ such that $\alpha(\sqrt[3]{2}) = \sqrt[3]{2}$ and $\alpha(\sqrt[3]{\beta}) = -\sqrt[3]{\beta}$. Find the fixed field of α .
10. Find the index $[\mathbb{Q}(\sqrt{f}) : \mathbb{Q}]$.
11. List the elements of the Galois group $G(\mathbb{Q}(\sqrt{r})/\mathbb{Q})$.
12. Describe an extension K of \mathbb{Q} such that $G(K/\mathbb{Q})$ is of order 3.
13. Verify whether $x^3 - 1$ is a cyclotomic polynomial.
14. Show that $x^3 - 1$ is solvable by radicals over \mathbb{Q} .

(14 x 1 = 14 weightage)

Turn over

Part B

*Answer any seven questions.
Each question carries 2 weightage.*

15. Show that Z_{10}/I is a field where I is the ideal generated by 5 in Z_{10}
16. Verify whether the field C of complex numbers is an algebraic extension of the field CP of rational numbers.
17. Let a be an algebraic element over a field F and $p \in F(a)$. Show that 3 is algebraic over F .
18. Let a be algebraic over a field E . Verify whether a is algebraic over every subfield F of E .
19. Show that if F is a finite field of characteristic p then F is an extension of Z .
20. Let E be an extension of a field F and $a \in E$ be algebraic over F . Let n be an automorphism of E leaving F fixed and $u(a) = 13$. Show that $irr(a; F) = irr(13; F)$.
21. Show that $Q(\bar{a})$ is a splitting field over Q .
22. Find all automorphisms of the field $Q(i)$ where i is square root of -1 .
23. Let K be the splitting field of $(x^2 - 2)(x^2 + 2)$ over Q . Describe the Galois group $G(K/Q)$.
24. Show that a regular 7-gon is not constructible by straight edge and compass.
(7 x 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries 4 weightage.*

25. (a) Show that if F is a field then the polynomial ring $F[x]$ is a principal ideal ring.
(b) Show that an ideal $(p(x))$ in $F[x]$ is maximal if and only if $p(x)$ is irreducible.
26. Let E be an extension of a field F and $a \in E$ be algebraic over F . Show that
 - (a) There exists an irreducible polynomial $p(x) \in F[x]$ such that $p(a) = 0$.
 - (b) If $q(x)$ is also an irreducible polynomial such that $q(a) = 0$ then $p(x)$ and $q(x)$ have same degree.
27. (a) Let F be a finite field of q elements. Show that $q = p^n$ for some natural number n where p is the characteristic of F .
(b) Show that if F and F' are finite fields having the same number of elements then F and F' are isomorphic.
28. (a) Define the n th cyclotomic polynomial $(D)_n(x)$ over a field F .
(b) Show that the Galois group of the 5th cyclotomic polynomial over Q is a cyclic group of order 4.
(2 x 4 = 8 weightage)