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Name.....

Reg. No.....

**SECOND SEMESTER M.Sc. DEGREE EXAMINATION, AUGUST 2013**

(CUCSS)

Mathematics

MT 2C 06—ALGEBRA II

Time : Three Hours

Maximum : 36 Weightage

**Part A**

Short answer questions (1-14)

Answer **all** questions. Each question has **one weightage**.

- (1) Find all  $c$  such that  $\mathbb{Z}_c[x]/(x^2 + cx + 1)$  is a field.
- (2) Define simple extension and give an example of it.
- (3) Find a subfield  $F$  of  $\mathbb{R}$  such that it is algebraic of degree 3 over  $\mathbb{Q}$ .
- (4) Find a basis of  $\mathbb{Q}(\sqrt{2}, \sqrt{3})$  over  $\mathbb{Q}$ .
- (5) Prove that squaring the circle is impossible.
- (6) Does there exist a field of 60 elements? Justify your answer.
- (7) Let  $F, E$  and  $K$  be fields such that  $F \subseteq E \subseteq K$ . Prove that  $G(K/E) \leq G(K/F)$ .
- (8) Let  $F$  be a field and let  $\alpha \in E$ . Prove that  $\alpha$  is separable over  $F$  if and only if  $\text{irr}(\alpha, F)$  has all zeros of multiplicity 1.
- (9) Prove that a finite extension  $E$  of a finite field  $F$  is a simple extension of  $F$ .
- (10) Let  $\sigma$  be the automorphism of  $\mathbb{Q}(\pi)$  that maps  $\pi$  to  $-\pi$ . Describe the fixed field of  $\sigma$ .
- (11) Give an example of two finite normal extensions  $K_1$  and  $K_2$  of the same field  $F$  such that  $K_1$  and  $K_2$  are not isomorphic fields but  $G(K_1/F) \cong G(K_2/F)$ .
- (12) How many elements are there in the splitting field of  $x^6 - 1$  over  $\mathbb{Z}_3$ .
- (13) Find  $1_3(x)$  over  $\mathbb{Z}_2$ .
- (14) Is splitting field of  $x^3 - 5$  over  $\mathbb{Q}$  a solvable Galois group? Justify your answer.

(14 x 1 = 14 weightage)

Turn over

**Part B**

Answer any seven from the following ten questions (15-24).

Each question has weightage 2.

- (15) Prove that a field has no proper nontrivial ideals.
- (16) Prove that  $\mathbb{Q}(2^{1/2}, 2^{1/3}) = \mathbb{Q}(2^{1/6})$ .
- (17) Prove that the set of algebraic numbers form a field.
- (18) Prove that there exists a finite field  $\mathbb{GF}(p^n)$  of  $p^n$  elements for every prime power  $p^n$ .
- (19) Show that for a prime  $p$ , the splitting field over  $\mathbb{Q}$  of  $x^p - 1$  is of degree  $p - 1$  over  $\mathbb{Q}$ .
- (20) Prove that the complex zeros of polynomials with real coefficients occur in conjugate pairs.
- (21) Prove that if  $E$  is an algebraic extension of a perfect field  $F$ , then  $E$  is perfect.
- (22) Prove that a finite extension of a field of characteristic zero is a simple extension.
- (23) Prove that the Galois group of a finite extension of a finite field is abelian.
- (24) Prove that the Galois group of the  $n$ th cyclotomic extension of  $\mathbb{Q}$  has  $\varphi(n)$  elements.

(7 x 2 = 14 weightage)

**Part C**

Answer any two from the following four questions (25-28).

Each question has weightage 4.

- (25) Let  $R$  be a commutative ring with unity. Prove that  $M$  is a maximal ideal of  $R$  if and only if  $R/M$  is a field.
- (26) Prove that every field  $F$  has an algebraic closure.
- (27) Let  $F$  and  $E$  be fields such that  $F \subset E \subset \bar{F}$ . Prove that  $E$  is a splitting field over  $F$  if and only if every automorphism of  $\bar{F}$  leaving  $F$  fixed maps  $E$  onto itself and induces an automorphism of  $E$  leaving  $F$  fixed.
- (28) Let  $F$  be a field of characteristic 0 and let  $a \in F$ . If  $K$  is a splitting field of  $x^n - a$  over  $F$ , then prove that  $G(K/F)$  is a solvable group.

(2 x 4 = 8 weightage)