

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2016

(CUCSS)

Mathematics

MT 1C 03—REAL ANALYSIS—I

(2016 Admissions)

Time : Three Hours

Maximum : 36 **Weightage**

Part A (Short Answer Questions)

*Answer all questions.**Each question has **weightage** 1.*

- Let X be a metric space. Define neighbourhood of $p \in X$ and prove that it is an open set.
- Construct a bounded set of real numbers with exactly two limit points.
- Let $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$. Show that E is closed and bounded in \mathbb{Q} . Is E compact.
- Let $f : [0, 2\pi) \rightarrow \{(\cos t, \sin t) : t \in [0, 2\pi)\}$ be defined by $f(t) = (\cos t, \sin t)$. Verify whether f is a **homeomorphism**.
- Identify the type of discontinuity at rational points $x = \frac{m}{n}$ (where m and n are integers without **any** common divisors) of the function f defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{m}{n} \\ n & \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
- Let f be defined on $[0, 2]$ by $f(x) = \frac{x}{x+1}$. If f uniformly continuous. Justify your answer.
- If f is a real differentiable function defined on $[a, b]$ and $f'(a) < c < f'(b)$, prove that there is a point $x \in (a, b)$ such that $f'(x) = c$.
- Prove that $\lim_{x \rightarrow 0} \sin x = 0$.

Turn over

9. Explain whether the Mean Value theorem is applicable to $f(x) = 2 + (x - 1)^2$ in $[0, 2]$.
10. If f is bounded on $[a, b]$, f is continuous at s where $a < s < b$ and $\mathbf{I}(x) = I(x - s)$ where I is the unit step function defined by $\mathbf{I}(x) = 0$ if $x < 0$ and $\mathbf{I}(x) = 1$ if $x > 0$, then prove that $\int_a^b f(x) \mathbf{I}(x - s) dx = f(s)$.
11. If f is continuous on $[a, b]$ and f has a continuous derivative on $[a, b]$ then prove that $\int_a^b f(x) dx = \int_a^b f'(x) dx$.
12. Define a rectifiable curve. Give an example of a rectifiable curve.
13. Let $\{f_n\}$ be a sequence of functions differentiable on $[a, b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, prove that $\{f_n\}$ converges uniformly on $[a, b]$.
14. Define **equicontinuous** family of functions. State a sufficient condition for a family $\{f_n\}$ of continuous functions defined on a compact metric space K to be **equicontinuous** on K .
(14 x 1 = 14 weightage)

Part B

Answer any **seven** questions from the following ten questions.
Each question has **weightage 2**.

15. Prove that the interior of a set S is the largest open subset of S .
16. Define a compact set. Explain with details why $S = (0, 1)$ is not compact.
17. If F is closed and K is compact prove that $F \cap K$ is compact.
18. If $f : X \rightarrow Y$ is continuous where X is a compact metric space and Y is a metric space, prove that f is uniformly continuous on X .
19. Prove that monotonic functions have no discontinuities of the second kind.
20. State Taylor's theorem. Illustrate with an example.
21. If f is bounded in $[a, b]$, f has only finitely many points of discontinuities on $[a, b]$ and f is continuous at every point at which f is discontinuous, prove that $f \in \mathcal{R}(a)$ on $[a, b]$.

22. If f is Riemann integrable on $[a, b]$ and if there exists a differentiable function F such that $F' = f$, prove that $\int_a^b f(x) dx = F(b) - F(a)$.
23. If $\{f_n\}$ is a sequence of continuous functions defined on E in a metric space and if $f_n \rightarrow f$ uniformly on E prove that f is continuous on E .
24. If $f_n \rightarrow f$ uniformly on $[a, b]$, $f_n \in \mathcal{R}(a)$ on $[a, b]$ where a is monotonically increasing on $[a, b]$, prove that $f \in \mathcal{R}(a)$ on $[a, b]$ and $\int_a^b f da = \lim_{n \rightarrow \infty} \int_a^b f_n da$.

(7 x 2 = 14 weightage)

Part C

Answer any two questions from the following four questions.
Each question has weightage 4.

25. (a) Let X be a compact set in \mathbb{R} and $A \subset X$ be closed. Prove that A is compact.
- (b) Let X and Y be metric spaces. Prove that $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y .
26. Let a increases monotonically and $a' \in \mathcal{R}$ on $[a, b]$. If f is a bounded real function on $[a, b]$ prove that $f \in \mathcal{R}(a)$ on $[a, b]$ if and only if $fa' \in \mathcal{R}$. Also prove that $\int_a^b f da = \int_a^b f(x) a'(x) dx$.
27. Prove that there exists a real continuous function on the real line which is nowhere differentiable.
28. State and prove the Stone-Weierstrass theorem.

(2 x 4 = 8 weightage)