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Name...............................

Reg. No...........................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2016

(CUCSS)

Mathematics

MT 10 03—REAL ANALYSIS--I

(2010 Admissions)

Time : Three Hours

Maximum : 36 Weightage

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question has 1 weightage.

1. Define convex set. Give an example of a non-convex set in R2.

2. For $x, y \in R^1$, let $d(x, y) = |x - 2y|$. Is d a metric on R^1? Justify your answer.

3. Is arbitrary intersection of closed sets closed? Justify your answer.

4. Let E be an infinite subset of a compact set K. Prove that E has a limit point in K.

5. Let f be a real uniformly continuous function on a bounded set E in R1. Prove that f is bounded on E.

6. Let f and g be differentiable functions on *(a, b)*. Prove that fg is differentiable on (a, *b)* and $(fg)'(x) = f'(x) g(x) + f(x) g'(x)$ for all $x \in (a, b)$.

7. Evaluate $\lim \log x$

8. Let a be an increasing function on [a, b], $a \le x_0 \le b$, a be a continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in R$ (a) on $[a, b]$ and $\int_a^b f \, d\alpha = 0$.

9. If $f_1 \in R$ (a) and $f_2 \in R$ (a) on *[a, b]*, then prove that $f_1 + f_2 \in R$ (a) on [a, b] and $\int_a (f_1 + f_2) \, d\alpha = \int_a f_1 \, d\alpha + \int_a f_2 \, d\alpha$.

10. Let y be a curve in the complex plane, defined on $[0, 2\pi]$ by $y(t) = e^{it}$. Find the length of y.

11. For n = 1, 2, 3,... and for real x, let $f_n(x) = \dfrac{x}{1 + nx^2}$. Show that $\{f_n\}$ is uniformly convergent.

12. Let $\{f_n\}$ be a sequence of real valued differentiable functions that converges to f. Is it true that $f'_n \to f'$? Justify your answer.

13. Define equicontinuity. Prove that functions in an equicontinuous family are continuous.

14. State Stone-Weierstrass theorem.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *from the following ten questions.*
Each question has weightage 2.

15. Prove that every infinite subset of a countable set A is countable.

16. Let X be a set. For $p, q \in X$, define :

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that d is a metric on X. Which subsets of the resulting metric space are open.

17. Let E be a closed set of real numbers which is bounded above. If y is the least upper bound of E, then prove that $y \in E$.

18. Let $[x]$ be the largest integer less than or equal to x. What type of discontinuities does the function $[x]$ have ?

19. Let f be a function defined on [a, b]. If f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then prove that $f'(x) = 0$.

20. If f is differentiable on [a, b], then prove that f' cannot have any simple discontinuities on $[a, b]$.

21. Let f be a bounded real function on [a, b] and a be monotonically increasing on [a, b]. If P' is a refinement of the partition P, then prove that :

$L(P, f, a) \le L(P', f, a)$.

22. Let $f \in R(a)$ on $[a, b]$. Prove that $|f| \in R(a)$ on $[a, b]$ and $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$.

23. Prove that the series :

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval.

24. For $n = 1, 2, \ldots$, let $f_n(x) = \dfrac{x}{x^2 + (1 - nx)^2}$. Show that $\{f_n\}$ is uniformly bounded on $[0, 1]$. Also

prove that $\{f_n\}$ is not equicontinuous on $[0, 1]$.

(7 x 2 = **14 weightage**)

Part C

Answer any two from the following four questions.
Each question has weightage 4.

25. (a) Prove that a set E is open if only if its complement is closed.

   (b) Prove that compact subsets of a metric space are closed.

26. State and prove Taylor's theorem.

27. (a) Let f be a bounded function on $[a, b]$. Prove that $f \in R(a)$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that :

   $$U(P, f, a) - L(P, f, a) < \varepsilon .$$

   (b) Let f be a bounded function and a be a monotonic increasing function on $[a, b]$. If f_1, f_2 are Riemann-Stieltjes integrable with respect to a on $[a, b]$, then prove that $f_1 + f_2$ is Riemann-Stieltjes integrable with respect to a on $[a, b]$.

28. Let $\{f_n\}$ be a sequence of continuous functions on a set E such that $f_n \to f$ uniformly on E. Prove that f is continuous on E.

(2 x 4 = 8 **weightage**)