

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CUCSS)

Mathematics

MT 1C 04—ODE AND CALCULUS OF VARIATIONS

Time : Three Hours

Maximum : 36 Weightage

Part A

Answer **all** questions.Each question carries 1 *weightage*.

1. Find a power series solution of the form $\sum a_n x^n$ of the differential equation $y' + y = 1$.
2. Determine the nature of the point $x = -1$ for the equation $x^2 (x^2 - 1)^2 y'' - 2(x - 1)y' + 3xy = 0$.
3. Find the indicial equation and its roots of the equation $x^3 + (\cos 2x - 1)y'' + 2xy' = 0$.
4. Find the general solution of the equation $(2x^2 + 2x)y'' + (1 + 5x)y' + y = 0$ near the singular point $x = 0$.
5. Find the first two terms of the Legendre series of the function

$$f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \end{cases}$$
6. Prove that for an integer $m \geq 0$, $J_{-m}(x)$ and $J_m(x)$ are linearly dependent.
7. Prove that $\frac{d}{dx} [x^m J_{\nu-1}(x)] = x^m J_{\nu-1}(x)$.
8. Describe the phase portrait of the system $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -y$.
9. Find the critical points of the non-linear system :

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = 2x^2 - y^2$$

Turn over

10. Show that $(0, 0)$ is an asymptotically stable critical point of the system :

$$\frac{dx}{dt} = -3x^3, \quad \frac{dy}{dt} = x^3 - 2y^3$$

11. Find the normal form of Bessel's equation $x^2 y'' + (x^2 - p^2) y = 0$.
12. State Sturm comparison theorem.
13. Show that $f(x, y) = xy$ satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$.

14. Find the extremal for the integral $\int_a^b (y'^2 - y^2) dx$.

(14 x 1 = 14 weightage)

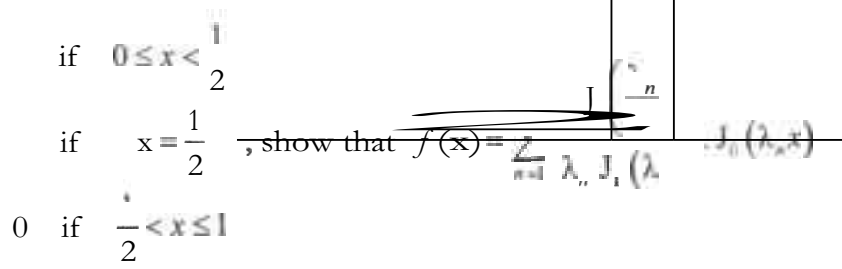
Part B

Answer any **seven** questions.
Each question carries 2 weightage.

15. Find the series solution $y(x)$ of the differential equation $y'' + xy = 0$ satisfying the condition $y(0) = 0, y'(0) = 1$.
16. Find the only Frobenius series solution of the equation $x^2 y'' - 3xy' + (4x + 4)y = 0$.
17. Determine all the regular singular points of the hypergeometric equation $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$.
18. Show that if $p_n(x)$ is defined by $p_n(x) = \frac{1}{n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n$, then $p_n(x)$ satisfies the Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a non-negative integer.
19. Show that between any two positive zeros of $J_0(x)$ there is a zero of $J_1(x)$ and that between any two positive zeros of $J_1(x)$ there is a zero of $J_0(x)$.

20. If $f(x)$ is defined by $f(x) =$

if $0 \leq x < \frac{1}{2}$ if $x = \frac{1}{2}$ 0 if $\frac{1}{2} < x \leq 1$, show that $f(x) = \sum_{n=1}^{\infty} \lambda_n J_1(\lambda_n x)$
--	---



where the λ_n 's are the positive zeros of $J_0(x)$.

21. Determine the nature and stability properties of the critical point $(0, 0)$ for the system :

$$\frac{dx}{dt} = 4x - 2y, \quad \frac{dy}{dt} = 5x + 2y.$$

22. Show that if $q(x) < 0$, and $u(x)$ is a nontrivial solution of $u'' + q(x)u = 0$, then $u(x)$ has at most one zero.

23. Using the method of Lagrange multipliers, show that the triangle with greatest area A for a given perimeter is equilateral.

24. Find the exact solution of the initial value problem $y' = 2x(1 + y)$, $y(0) = 0$. Starting with $y_0(x) = 0$, apply Picard's method to calculate $y_1(x)$, $y_2(x)$, $y_3(x)$ and compare these results with the exact solution.

(7 x 2 = 14 weightage)

Part C

*Answer any two questions.
Each question carries 4 weightage.*

25. Discuss the general solution of the hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

near the singular point $x = 0$.

26. State and prove the orthogonality property of the Legendre polynomials.

27. Find the general solution of the system :

$$\frac{dx}{dt} = 7x + 6y, \quad \frac{dy}{dt} = 2x + 6y.$$

28. Obtain Euler's differential equation for an extremal.

(2 x 4 = 8 weightage)