II 33334 (Pages : 3) Name

Reg. No..................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, FEBRUARY 2013

(CUCSS)

Mathematics

MT 1C 04—ODE AND CALCULUS OF VARIATIONS

(2010 admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Define radius of convergence of a power series $\sum a_n x^n$.
2. Determine the nature of the point x = 1 for the equation :

   $$x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0.$$

3. Find the indicial equation and its roots of the equation :

   $$4x^2 y'' + (2x^4 - 5x)y' + (3x^2 + 2)\,y = 0.$$

4. Evaluate : $\lim_{x \to \infty} F\left(a, a, \frac{3}{2}, \frac{-x^2}{4a^2}\right)$
5. Show that $P_{2n+1}(0) = 0$, where $P_n(x)$ is the Legendre polynomial of degree n.
6. Define gamma function and show that $\Gamma(p+1) = p\Gamma(p)$
7. Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$
8. Describe the phase portrait of the system : $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2$
9. Find the critical points of the non-linear system :

   $$\frac{dx}{dt} = y(x^2+1), \frac{dy}{dt} = 2xy^2.$$

10. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

• Turn over

11. Find the normal form of Bessel's equation $x^2 y \quad + xy \quad + (x^2 - p \quad) y = 0$, where p is a non-negative constant.

12. State sturm comparison theorem.

13. Show that $f(x, y) = x^2$ I y I satisfies a Lipschitz condition on the rectangle I x I **1** and $|y|$ **1.**

14. Find the stationary function of $\overset{4}{} \quad xy' \quad$ **(YT** dx which is determined by the boundary conditions y (0) = 0 and y (4) = 3.

(14 x 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 *weightage.*

15. Show that $\tan x = x + \frac{1}{3} x \quad + \frac{2}{15} x \quad + \ldots$ by solving y = **1+** y^2 ; y (0) = **0** in two ways.

16. Determine all the regular singular points of the hypergeometric equation :

$$x\,(1-x)\,y'' + [c-(a+b+1)x] \quad - aby = \mathbf{0}.$$

17. Let $f(x)$ be a function defined on the interval —1 < x S**1 and I** $= \int^{1} [f(x) \quad p(x)]^{2} dx,$ where $p(x)$ is a polynomial of degree n. Show that I is minimum when $p(x)$ is precisely the sum of the first (n + 1) terms of the Legendre series off (x).

18. Obtain J_p (x), the Bessel function of first kind.

19. Prove that the positive zeros of J_p *(x)* and J_{p+1} *(x)* occur alternately, in the sense that between each pair of consecutive positive zeros of either there is exactly one zero of the other.

20. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = 3x + 4y\,;\ \frac{dy}{dt} = -2x + 3y$$

21. Show that if there exists a Liapunov function E (x, y) for the system :

$\frac{dx}{dt} = F(x, y)\,;\ \frac{dy}{dt} = G(x, y)$ then the critical point (0, 0) is stable.

22. Let $u(x)$ be any non-trivial solution of $u'' + q(x)\,u = \mathbf{0},$ where $q(x) > 0$ for all $x > 0$. Show that if

$\int q(x)\,dx = \infty,$ then u (x) has infinitely many zeros on the positive x-axis.

23. Show that the eigen functions of the boundary. value problem $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + 2\,q(x)\,y = 0$;

    $y\,(a) = y\,(b) = 0$ satisfy the relation $\int_a^b q\, y_m(x)\, y_n(x)\, dx = 0$ is $m \neq n$.

24. A curve in the first quadrant joins (0, **0)** and **(1, 0)** and has a given area beneath it. Show that the shortest such curve is an arc of a circle.

(7 x 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries **4** *weightage.*

25. Find two independent Frobenius series solutions of the equation $xy'' + 2y' + xy = 0$.

26. Derive Rodrigue's formula for the Legendre polynomials and use it to find $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$.

27. Find the general solution of the system : $\frac{dx}{dt} = 5x + 4y$; $\frac{dy}{dt} = -x + y$

28. Explain Picard's method of successive approximations to solve the initial value problem $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$, where $f(x, y)$ is an arbitrary function defined and continuous in some neighbourhood of the point (x_0, y_0). Use this method to calculate $y_1(x)$, $y_2(x)$, $y_3(x)$ starting with $y_0(x) = 0$ for the initial value problem $y' = 2x(1 + y)$, $y(0) = 0$.

(2 x 4 = 8 weightage)