D 13190 (Pages : 3) Name..................................

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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2016

(CUCSS)

Mathematics

MT 1C 04—NUMBER THEORY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Define the Mobius function $\mu(n)$ and the Euler totient function +(n).
2. If f is multiplicative, prove that $f(1) = 1$.
3. If f and g are arithmetical functions, prove that :

   $(f * g)' = f' * g + f * g'$

   where f' denotes the derivative off.
4. Prove that [24— 2 [x] is either 0 or 1.
5. For $x Z 1$, prove that

   $$E_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] = \log [x]!$$

6. State Abel's identity.
7. For x 2, prove that $\pi(x) = \frac{(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt.$
8. If a > 0 and b > 0, then show that

   $$\lim_{x \to \infty} \frac{\pi(ax)}{\pi(bx)} = \frac{a}{b}$$

9. Prove that $\lim \left(\frac{M(x)}{x} - \frac{H(x)}{x \log x} \right) = 0$

Turn over

10. If p is an odd prime, prove that $\sum_{r=1} (r/p) = 0$.

11. Evaluate the Legendre symbol $\left(3/383\right)$.

12. If P is an odd positive integer, prove that $\left(-1/P\right) = (-1)^{(P}$

13. Prove that the product of two shift enciphering transformations is also a shift enciphering transformation.

14. Fine the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$ mod 5.

(14 x 1 = 14 weightage)

Part B

Answer any seven *questions.*
Each question carries a weightage of 2.

15. Let f be multiplicative. Prove that f is completely multiplicative if and only if :

$f^{-1}(n) = \mu(n) f(n)$ for all n 1.

16. Prove that $\prod_{t/n} t = n^{d(n)/2}$ where $d(n)$ denotes the number of positive divisors of n.

17. If x 1, prove that $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + G(s) + O(x^{-s})$ if $s > 0$, $s \neq 1$.

18. State and prove Legendre's identilty.

19. Prove that the following two relations are equivalent.

(a) $\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log x}\right)$.

(b) $\vartheta(x) = x + O\left(\frac{x}{\log x}\right)$.

20. If $\{a(n)\}$ is a non-negative sequence such that :

$\sum_{n \leq x} a(n)\left[\frac{x}{n}\right] = x \log x + O(x)$ for all x ≥ 1, then prove that there is a constant A > 0 and an $x_0 > 0$

such that $\sum_{n \leq x} a(n)$ Ax for all x x_0.

21. If $A(x) = \sum_{n \le x} \frac{[illegible]}{n}$, then prove that the relation $A(x) = 0(1)$ as $x \to \infty$ implies the prime number theorem.

22. Prove that the Legendre's symbol *(nip) is* a completely multiplicative function of n.

23. Explain briefly about digraph transformations.

24. How will you authenticate a message in public key cryptosystem.

(7 x 2 = 14 weightage)

Part C

Answer any two questions.
Each question carries a weightage of 4.

25. Prove that the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group under Dirichlet product.

26. For $n \geq 1$ prove that the nth prime p_n satisfies the inequalities

$$[illegible]\, n \log c < p_n < 12\left(n \log n + n \log \frac{12}{e}\right).$$

27. Determine those odd primes p for which 3 is a quadratic residue mod p and those for which it is a non-residue.

28. Suppose that the following 40-letter alphabet is used for all plaintexts and cipher texts : A-Z with numerical equivalents 0-25, blank = 26, = 27, ? = 28, \$ = 29, the numerals 0-9 with numerical equivalents 30-39. Suppose that plaintext message units are digraphs and cipher text message units are trigraphs.

(ie, $k = 2$, $l = 3$, $40^2 < n_A < 40^3$ for all n_A).

(a) Send the message "SEND \$ 7500" to a user whose enciphering key is $(C_A, e_A) = (2047, 179)$.

(b) Break the code by factoring n_A and then compute the deciphering key (B_A, d_A).

(2 x 4 = 8 weightage)