C 1249 (Pages : 3) Name..................................

Reg. No..................................

SIXTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION, MARCH 2021

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. State Division algorithm.
2. Find $gcd(272,1479)$.
3. Show that the Diophantine equation $14x + 35y = 651$ has an integer solution.
4. Define e-prime. Give an example of a prime number.
5. Find the last two digits of the number 9^{9^9}.
6. State Fermat's theorem.
7. Find $\tau(180)$.
8. Define subspace of a vector space.
9. Give a basis for $Mat_{2\times 2}(\mathbb{R})$.
10. What do you mean by a Linear Transformation ?
11. Give an injective linear map which is not surjective.
12. State the Dimension theorem.

(12 × 1 = 12 marks)

Section B

Answer at least **eight** *questions.*
Each question carries 6 *marks.*
All questions can be attended.
Overall Ceiling 48.

13. Show that the expression $\frac{a(a^2+2)}{3}$ is an integer for al. $a \geq 1$.
14. If $a|bc$, with $gcd(a, b) = 1$, prove that $a|c$.

Turn over

15. Use the Euclidean Algorithm to obtain integers x and y satisfying $gcd(119, 272) = 119x + 272y$.

16. Prove that the number $\sqrt{5}$ is irrational.

17. Find the canonical form of 2093.

18. If p_n is the n^{th} prime number, prove that $p_n \leq 2^{2n-1}$.

19. Find the remainder obtained upon dividing the sum $1! + 2! + 3! + \ldots + 99! + 100!$ by 12.

20. Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a polynomial function of x with integral coefficients c_k. If $a \equiv b \pmod{n}$, prove that $P(a) \equiv P(b) \pmod{n}$.

21. If $(n - 1)! \equiv -1 \pmod{n}$, then prove that n must be prime.

22. Prove that every plane through the origin is a subspace of $\mathbb{R}^3$.

23. Show that $\{(1, 1, 0, 0), (-1, -1, 1, 2), (1, -1, 1, 3), (0, 1, -1, -3)\}$ is a basis of $\mathbb{R}^4$.

24. Let S be a subset of the vector space V. Prove that S is basis if and only if S is a minimal spanning set.

25. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map given by $f(a, b, c, d) = (a + b, b - c, a + d)$. Find $Im\, f$ and a basis for $Im\, f$.

26. Prove that if the linear mapping $f : V \longrightarrow W$ is injective and $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent subset of V then $\{f(v_1), f(v_2), \ldots, f(v_n)\}$ is a linearly independent subset of W.

(8 × 6 = 48 marks)

Section C

Answer at least **five** *questions.*
Each question carries 9 *marks.*
All questions can be attended.
Overall Ceiling 45.

27. Let a and b be integers, not both zero. Prove that a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.

28. Prove that if $k > 0$, then $gcd(ka, kb) = k\,gcd(a, b)$.

29. State and Prove Fundamental Theorem of Arithmetic.

30. Prove that the system of linear congruences $ax + by \equiv r \pmod{n}$; $cx + dy \equiv s \pmod{n}$ has a unique solution modulo n whenever $gcd(ad - bc, n) = 1$.

31. If n is an odd pseudoprime, prove that $M_n = 2^n - 1$ is a larger one.

32. Let V be a vector space that is spanned by the finite set $G = \{v_1, v_2, v_3, ..., v_n\}$. If $I = \{w_1, w_2, ..., w_m\}$ is linearly independent subset of V then show that $m \leq n$.

33. Let V be finite dimensional vector space. If G is a finite spanning set of V and if I is a linearly independent subset of V such that $I \subseteq G$ then prove that there is a basis B of V such that $I \subseteq B \subseteq G$.

34. Prove that a linear mapping is completely and uniquely determined by its action on basis.

35. If $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map such that $f(1, 1, 0) = (1, 2)$, $f(1, 0, 1) = (0, 0)$, $f(0, 1, 1) = (2, 1)$, then find $f(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$.

(5 × 9 = 45 marks)

Section D

Answer any **one** *question.*
The question carries 15 *marks.*

36. a) Let a and b be integers, not both zero. For positive integer d, prove that $d = gcd(a, b)$ if and only if

    (i) $d|a$ and $d|b$

    (ii) Whenever $c|a$ and $c|b$, then $c|d$.

    b) A customer bought a dozen pieces of fruit, apples and oranges, for $1.32. If an apple costs 3 cents more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought.

37. State and prove Chinese Remainder Theorem.

38. Let V and W be vector spaces each of dimension n over a field F. If $f : V \rightarrow W$ is linear then prove that the following statements are equivalent :

    (i) f is injective ;

    (ii) f is surjective ;

    (iii) f is bijective ;

    (iv) f carries bases to bases.

(1 × 15 = 15 marks)